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# A sequence of Albin type continuous martingales with Brownian marginals and scaling

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*Dedicated to Lester Dubins (1921-2010) to whom the third author owes a lot.*

**Summary.** Closely inspired by Albin's method which relies ultimately on the duplication formula for the Gamma function, we exploit Gauss' multiplication formula to construct a sequence of continuous martingales with Brownian marginals and scaling.

**Key words:** Martingales, Brownian marginals

## 1 Motivation and main results

(1.1) Knowing the law of a "real world" random phenomena, i.e. random process,  $(X_t, t \geq 0)$  is often extremely difficult and in most instances, one avails only of the knowledge of the 1-dimensional marginals of  $(X_t, t \geq 0)$ . However, there may be many different processes with the same given 1-dimensional marginals.

In the present paper, we make explicit a sequence of continuous martingales  $(M_m(t), t \geq 0)$  indexed by  $m \in \mathbb{N}$  such that for each  $m$ ,

i)  $(M_m(t), t \geq 0)$  enjoys the Brownian scaling property: for any  $c > 0$ ,

$$(M_m(c^2t), t \geq 0) \stackrel{(law)}{=} (cM_m(t), t \geq 0)$$

ii)  $M_m(1)$  is standard Gaussian.

Note that, combining i) and ii), we get, for any  $t > 0$

$$M_m(t) \stackrel{(law)}{=} B_t,$$

where  $(B_t, t \geq 0)$  is a Brownian motion, i.e.  $M_m$  admits the same 1-dimensional marginals as Brownian motion.

**(1.2)** Our main result is the following extension of Albin's construction [1] from  $m = 1$  to any integer  $m$ .

**Theorem 1.** *Let  $m \in \mathbb{N}$ . Then, there exists a continuous martingale  $(M_m(t), t \geq 0)$  which enjoys i) and ii) and is defined as follows:*

$$M_m(t) = X_t^{(1)} \dots X_t^{(m+1)} Z_m \quad (1)$$

where  $(X_t^{(i)}, t \geq 0)$ , for  $i = 1, \dots, m+1$ , are independent copies of the solution of the SDE

$$dX_t = \frac{1}{m+1} \frac{dB_t}{X_t^m}; \quad X_0 = 0 \quad (2)$$

and, furthermore,  $Z_m$  is independent from  $(X^{(1)}, \dots, X^{(m+1)})$  and

$$Z_m \stackrel{(law)}{=} (m+1)^{1/2} \left( \prod_{j=0}^{m-1} \beta\left(\frac{1+2j}{2(m+1)}, \frac{m-j}{m+1}\right) \right)^{\frac{1}{2(m+1)}} \quad (3)$$

where  $\beta(a, b)$  denotes a beta variable with parameter  $(a, b)$  with density

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1} 1_{[0,1]}(x)$$

and the beta variables on the right-hand side of (3) are independent.

**Remark:** For  $m = 1$ ,  $Z_1 = \sqrt{2} \left( \beta(\frac{1}{4}, \frac{1}{2}) \right)^{1/4}$  and we recover the distribution of  $Y := Z_1$  given by (2) in [1].

**(1.3)** For the convenience of the reader, we also recall that, if one drops the continuity assumption when searching for martingales  $(M(t); t \geq 0)$  satisfying i) and ii), then, the Madan-Yor construction [5] based on the "Azéma-Yor under scaling" method provides such a martingale.

Precisely, starting from a Brownian motion  $(B_u, u \geq 0)$  and denoting  $S_u = \sup_{s \leq u} B_s$ , introduce the family of stopping times

$$\tau_t = \inf\{u, S_u \geq \psi_t(B_u)\}$$

where  $\psi_t$  denotes the Hardy-Littlewood function associated with the centered Gaussian distribution  $\mu_t$  with variance  $t$ , i.e.

$$\begin{aligned} \psi_t(x) &= \frac{1}{\mu_t([x, \infty[)} \int_x^\infty y \exp\left(-\frac{y^2}{2t}\right) \frac{dy}{\sqrt{2\pi t}} \\ &= \sqrt{t} \exp\left(-\frac{x^2}{2t}\right) / \mathcal{N}(x/\sqrt{t}) \end{aligned}$$

where  $\mathcal{N}(a) = \int_a^\infty \exp(-\frac{y^2}{2}) dy$ . Then,  $M_t = B_{\tau_t}$  is a martingale with Brownian marginals.

Another solution has been given by Hamza and Klebaner [4].

## 2 Proof of the theorem

**Step 1:** For  $m \in \mathbb{R}$  and  $c \in \mathbb{R}$ , we consider the stochastic equation:

$$dX_t = c \frac{dB_t}{X_t^m}, \quad X_0 = 0.$$

This equation has a unique weak solution which can be defined as a time-changed Brownian motion

$$(X_t) \stackrel{(law)}{=} W(\alpha^{(-1)}(t))$$

where  $W$  is a Brownian motion starting from 0 and  $\alpha^{(-1)}$  is the (continuous) inverse of the increasing process

$$\alpha(t) = \frac{1}{c^2} \int_0^t W_u^{2m} du.$$

We look for  $k \in \mathbb{N}$  and  $c$  such that  $(X_t^{2k}, t \geq 0)$  is a squared Bessel process of some dimension  $d$ . It turns out, by application of Itô's formula, that we need to take  $k = m + 1$  and  $c = \frac{1}{m+1}$ . Thus, we find that  $(X_t^{2(m+1)}, t \geq 0)$  is a squared Bessel process with dimension  $d = k(2k - 1)c^2 = \frac{2m+1}{m+1}$ . Note that the law of a BESQ( $d$ ) process at time 1 is well known to be that of  $2\gamma_{d/2}$ , where  $\gamma_a$  denotes a gamma variable with parameter  $a$ . Thus, we have:

$$|X_1| \stackrel{(law)}{=} \left( 2\gamma_{\frac{2m+1}{2(m+1)}} \right)^{\frac{1}{2(m+1)}} \quad (4)$$

**Step 2:** We now discuss the scaling property of the solution of (2). From the scaling property of Brownian motion, it is easily shown that, for any  $\lambda > 0$ , we get:

$$(X_{\lambda t}, t \geq 0) \stackrel{(law)}{=} (\lambda^\alpha X_t, t \geq 0)$$

with  $\alpha = \frac{1}{2(m+1)}$ , that is, the process  $(X_t, t \geq 0)$  enjoys the scaling property of order  $\frac{1}{2(m+1)}$ .

**Step 3:** Consequently, if we multiply  $m + 1$  independent copies of the process  $(X_t, t \geq 0)$  solution of (2), we get a process

$$Y_t = X_t^{(1)} \dots X_t^{(m+1)}$$

which is a martingale and has the scaling property of order  $\frac{1}{2}$ .

**Step 4:** Finally, it suffices to find a random variable  $Z_m$  independent of the processes  $X_t^{(1)}, \dots, X_t^{(m+1)}$  and which satisfies:

$$N \stackrel{(law)}{=} X_1^{(1)} \dots X_1^{(m+1)} Z_m \quad (5)$$

where  $N$  denotes a standard Gaussian variable. Note that the distribution of any of the  $X_1^{(i)}$ 's is symmetric. We shall take  $Z_m \geq 0$ ; thus, the distribution of  $Z_m$  shall be determined by its Mellin transform  $\mathcal{M}(s)$ . From (5),  $\mathcal{M}(s)$  satisfies:

$$\mathbb{E}[(2\gamma_{1/2})^{s/2}] = \left( \mathbb{E}[(2\gamma_{d/2})^{s/2(m+1)}] \right)^{m+1} \mathcal{M}(s)$$

with  $d = \frac{2m+1}{m+1}$ , that is:

$$2^{s/2} \frac{\Gamma(\frac{1+s}{2})}{\Gamma(\frac{1}{2})} = 2^{s/2} \left( \frac{\Gamma(\frac{d}{2} + \frac{s}{2(m+1)})}{\Gamma(\frac{d}{2})} \right)^{m+1} \mathcal{M}(s)$$

that is precisely:

$$\frac{\Gamma(\frac{1+s}{2})}{\Gamma(\frac{1}{2})} = \left( \frac{\Gamma(\frac{2m+1+s}{2(m+1)})}{\Gamma(\frac{2m+1}{2(m+1)})} \right)^{m+1} \mathcal{M}(s). \quad (6)$$

Now, we recall Gauss multiplication formula ([2], see also [3])

$$\Gamma(kz) = \frac{k^{kz-1/2}}{(2\pi)^{\frac{k-1}{2}}} \prod_{j=0}^{k-1} \Gamma\left(z + \frac{j}{k}\right) \quad (7)$$

which we apply with  $k = m+1$  and  $z = \frac{1+s}{2(m+1)}$ . We then obtain, from (7)

$$\frac{\Gamma(\frac{1+s}{2})}{\sqrt{\pi}} = \frac{(m+1)^{s/2}}{(2\pi)^{m/2}} \frac{1}{\sqrt{\pi}} \prod_{j=0}^m \Gamma\left(\frac{1+s+2j}{2(m+1)}\right) \quad (8)$$

$$= (m+1)^{s/2} \prod_{j=0}^m \left( \frac{\Gamma(\frac{1+s+2j}{2(m+1)})}{\Gamma(\frac{1+2j}{2(m+1)})} \right) \quad (9)$$

since the two sides of (8) are equal to 1 for  $s = 0$ . We now plug (9) into (6) and obtain

$$(m+1)^{s/2} \prod_{j=0}^m \left( \frac{\Gamma(\frac{1+s+2j}{2(m+1)})}{\Gamma(\frac{1+2j}{2(m+1)})} \right) = \left( \frac{\Gamma(\frac{2m+1+s}{2(m+1)})}{\Gamma(\frac{2m+1}{2(m+1)})} \right)^{m+1} \mathcal{M}(s) \quad (10)$$

We note that for  $j = m$ , the same term appears on both sides of (10), thus (10) may be written as:

$$(m+1)^{s/2} \prod_{j=0}^{m-1} \left( \frac{\Gamma(\frac{1+s+2j}{2(m+1)})}{\Gamma(\frac{1+2j}{2(m+1)})} \right) = \left( \frac{\Gamma(\frac{2m+1+s}{2(m+1)})}{\Gamma(\frac{2m+1}{2(m+1)})} \right)^m \mathcal{M}(s) \quad (11)$$

In terms of independent gamma variables, the left-hand side of (11) equals:

$$(m+1)^{s/2} \mathbb{E} \left[ \left( \prod_{j=0}^{m-1} \gamma_{\frac{1+2j}{2(m+1)}}^{(j)} \right)^{\frac{s}{2(m+1)}} \right] \quad (12)$$

whereas the right-hand side of (11) equals:

$$\mathbb{E} \left[ \left( \prod_{j=0}^{m-1} \gamma_{\frac{1+2m}{2(m+1)}}^{(j)} \right)^{\frac{s}{2(m+1)}} \right] \mathcal{M}(s) \quad (13)$$

where the  $\gamma_{a_j}^{(j)}$  denote independent gamma variables with respective parameters  $a_j$ .

Now, from the beta-gamma algebra, we deduce, for any  $j \leq m-1$ :

$$\gamma_{\frac{1+2j}{2(m+1)}}^{(j)} \stackrel{(law)}{=} \gamma_{\frac{1+2m}{2(m+1)}}^{(j)} \beta\left(\frac{1+2j}{2(m+1)}, \frac{m-j}{m+1}\right).$$

Thus, we obtain, again by comparing (12) and (13):

$$\mathcal{M}(s) = (m+1)^{s/2} \mathbb{E} \left[ \left( \prod_{j=0}^{m-1} \beta\left(\frac{1+2j}{2(m+1)}, \frac{m-j}{m+1}\right) \right)^{\frac{s}{2(m+1)}} \right]$$

which entails:

$$\mathbb{E}[Z_m^s] = (m+1)^{s/2} \mathbb{E} \left[ \left( \prod_{j=0}^{m-1} \beta\left(\frac{1+2j}{2(m+1)}, \frac{m-j}{m+1}\right) \right)^{\frac{s}{2(m+1)}} \right]$$

that is, equivalently,

$$Z_m \stackrel{(law)}{=} (m+1)^{1/2} \left( \prod_{j=0}^{m-1} \beta\left(\frac{1+2j}{2(m+1)}, \frac{m-j}{m+1}\right) \right)^{\frac{1}{2(m+1)}}$$

### 3 Some remarks about Theorem 1

#### 3.1 A further extension

We tried to extend Theorem 1 by taking a product of independent martingales  $X^{(i)}$ , solution of (2) with different  $m_i$ 's. Here are the details of our attempt. We are looking for the existence of a variable  $Z$  such that the martingale

$$M(t) = \left( \prod_{j=0}^{p-1} X_t^{(m_j)} \right) Z$$

satisfies the properties i) and ii). Here  $p, (m_j)_{0 \leq j \leq p-1}$  are integers and  $X^{(m_j)}$  is the solution of the EDS (2) associated to  $m_j$ , the martingales being independent for  $j$  varying. In order that  $M$  enjoys the Brownian scaling property, we need the following relation

$$\sum_{j=0}^{p-1} \frac{1}{m_j + 1} = 1. \quad (14)$$

Following the previous computations, see (6), the Mellin transform  $\mathcal{M}(s)$  of  $Z$  should satisfy

$$\frac{\Gamma(\frac{1+s}{2})}{\Gamma(\frac{1}{2})} = \left( \prod_{j=0}^{p-1} \frac{\Gamma(\frac{2m_j+1+s}{2(m_j+1)})}{\Gamma(\frac{2m_j+1}{2(m_j+1)})} \right) \mathcal{M}(s). \quad (15)$$

We recall (see (9)) the Gauss multiplication formula

$$\frac{\Gamma(\frac{1+s}{2})}{\sqrt{\pi}} = p^{s/2} \prod_{j=0}^{p-1} \left( \frac{\Gamma(\frac{1+s+2j}{2p})}{\Gamma(\frac{1+2j}{2p})} \right) \quad (16)$$

To find  $\mathcal{M}(s)$  from (15), (16), we give some probabilistic interpretation:

$$\frac{\Gamma(\frac{1+s+2j}{2p})}{\Gamma(\frac{1+2j}{2p})} = \mathbb{E}[\gamma_{(1+2j)/2p}^{s/2p}]$$

whereas

$$\frac{\Gamma(\frac{2m_j+1+s}{2(m_j+1)})}{\Gamma(\frac{2m_j+1}{2(m_j+1)})} = \mathbb{E}[\gamma_{(1+2m_j)/2(m_j+1)}^{s/2(m_j+1)}].$$

Thus, we would like to factorize

$$\gamma_{(1+2j)/2p}^{1/2p} \stackrel{(law)}{=} \gamma_{(1+2m_j)/2(m_j+1)}^{1/2(m_j+1)} z_{m_j,p}^{(j)} \quad (17)$$

for some variable  $z_{m_j,p}^{(j)}$  to conclude that

$$Z = p^{1/2} \prod_{j=0}^{p-1} z_{m_j,p}^{(j)}.$$

It remains to find under which condition the identity (17) may be fulfilled. We write

$$\gamma_{(1+2j)/2p}^{1/2p} \stackrel{(law)}{=} \gamma_{(1+2m_j)/2(m_j+1)}^{p/(m_j+1)} (z_{m_j,p}^{(j)})^{2p}. \quad (18)$$

Now, if  $\frac{1+2j}{2p} < \frac{1+2m_j}{2(m_j+1)}$ , we may apply the beta-gamma algebra to obtain

$$\gamma_{(1+2j)/2p}^{(law)} \stackrel{=}{=} \gamma_{(1+2m_j)/2(m_j+1)} \beta\left(\frac{1+2j}{2p}, \frac{1+2m_j}{2(m_j+1)} - \frac{1+2j}{2p}\right)$$

but in (18), we need to have on the right-hand side  $\gamma_{(1+2m_j)/2(m_j+1)}^{p/(m_j+1)}$  instead of  $\gamma_{(1+2m_j)/2(m_j+1)}$ .

However, it is known that

$$\gamma_a^{(law)} \stackrel{=}{=} \gamma_a^c \gamma_{a,c}$$

for some variable  $\gamma_{a,c}$  independent of  $\gamma_a$  for any  $c \in (0, 1]$ . This follows from the self-decomposable character of  $\ln(\gamma_a)$ . Thus, we seem to need  $\frac{p}{m_j+1} \leq 1$ . But, this condition is not compatible with (14) unless  $m_j = m = p - 1$ .

### 3.2 Asymptotic study

We study the behavior of the product  $X_1^{(1)} \dots X_1^{(m+1)}$ , resp.  $Z_m$ , appearing in the right-hand side of the equality in law (5), when  $m \rightarrow \infty$ . Recall from (4) that

$$|X_1| \stackrel{(law)}{=} \left(2\gamma_{\frac{2m+1}{2(m+1)}}\right)^{\frac{1}{2(m+1)}}.$$

We are thus led to consider the product

$$\Theta_{a,b,c}^{(p)} = \left(\prod_{i=1}^p \gamma_{a-b/p}^{(i)}\right)^{c/p}$$

where in our set up of Theorem 1,  $p = m + 1$ ,  $a = 1$ ,  $b = c = 1/2$ .

$$\begin{aligned} \mathbb{E}[(\Theta_{a,b,c}^{(p)})^s] &= \prod_{i=1}^p \mathbb{E}[\left(\gamma_{a-b/p}^{(i)}\right)^{cs/p}] \\ &= \left(\frac{\Gamma(a - \frac{b}{p} + \frac{cs}{p})}{\Gamma(a - \frac{b}{p})}\right)^p \\ &= \exp[p(\ln(\Gamma(a + \frac{cs-b}{p})) - \ln(\Gamma(a - \frac{b}{p})))] \\ &\rightarrow \exp\left(\frac{\Gamma'(a)}{\Gamma(a)}cs\right). \end{aligned}$$

Thus, it follows that

$$\Theta_{a,b,c}^{(p)} \xrightarrow[p \rightarrow \infty]{\mathbb{P}} \exp\left(\frac{\Gamma'(a)}{\Gamma(a)}c\right),$$

implying that

$$|X_1^{(1)} \dots X_1^{(m+1)}| \xrightarrow[p \rightarrow \infty]{\mathbb{P}} \exp(-\gamma/2) \quad (19)$$

and



$$\exp(-\gamma/2) Z_m \xrightarrow[m \rightarrow \infty]{(law)} |N|. \quad (20)$$

where  $\gamma = -\Gamma'(1)$  is the Euler constant.

We now look for a central limit theorem for  $\Theta_{a,b,c}^{(p)}$ . We consider the limiting distribution of

$$\begin{aligned} & \sqrt{p} \left\{ \frac{c}{p} \sum_{i=1}^p \ln(\gamma_{a-b/p}^{(i)}) - c \frac{\Gamma'(a)}{\Gamma(a)} \right\}. \\ & \mathbb{E} \left( \exp \left[ cs \sqrt{p} \left\{ \frac{1}{p} \sum_{i=1}^p \ln(\gamma_{a-b/p}^{(i)}) - \frac{\Gamma'(a)}{\Gamma(a)} \right\} \right] \right) \\ &= \mathbb{E} \left[ \prod_{i=1}^p \left( \gamma_{a-b/p}^{(i)} \right)^{cs/\sqrt{p}} \right] \exp(-cs \sqrt{p} \frac{\Gamma'(a)}{\Gamma(a)}) \\ &= \mathbb{E} \left[ \left( \gamma_{a-b/p}^{(i)} \right)^{cs/\sqrt{p}} \right]^p \exp(-cs \sqrt{p} \frac{\Gamma'(a)}{\Gamma(a)}) \\ &= \left( \frac{\Gamma(a - \frac{b}{p} + \frac{cs}{\sqrt{p}})}{\Gamma(a - \frac{b}{p})} \right)^p \exp(-cs \sqrt{p} \frac{\Gamma'(a)}{\Gamma(a)}) \\ &= \exp[p(\ln(\Gamma(a - \frac{b}{p} + \frac{cs}{\sqrt{p}})) - \ln(\Gamma(a - \frac{b}{p}))) - cs \sqrt{p} \frac{\Gamma'(a)}{\Gamma(a)}] \\ &= \exp(\frac{c^2 s^2}{2} (\ln(\Gamma))''(a) + O(m^{-1/2})) \end{aligned}$$

We thus obtain that

$$\sqrt{p} \left\{ \frac{c}{m} \sum_{i=1}^m \ln(\gamma_{a-b/m}^{(i)}) - c \frac{\Gamma'(a)}{\Gamma(a)} \right\} \xrightarrow{(law)} N(0, \sigma^2) \quad (21)$$

where  $N(0, \sigma^2)$  denotes a centered Gaussian variable with variance:

$$\sigma^2 = c^2 (\ln(\Gamma))''(a) = c^2 \left[ \frac{\Gamma''(a)}{\Gamma(a)} - \left( \frac{\Gamma'(a)}{\Gamma(a)} \right)^2 \right].$$

or, equivalently

$$\left( \Theta_{a,b,c}^{(p)} \exp\left(\frac{\Gamma'(a)}{\Gamma(a)} c\right) \right)^{\sqrt{p}} \xrightarrow[p \rightarrow \infty]{(law)} \exp(N(0, c^2 (\ln(\Gamma))''(a))). \quad (22)$$

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